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# Fixed Point Theorem for α-ψ–Expansive Mappings in Digital Metric Spaces

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**Abstract—***Digital topology has attracted the attention of many researchers owing to its potential applications in some areas such as computer science, image processing, topology and fixed point theory. Recently some fixed point results have been appeared in digital metric spaces. In this paper, we prove some common fixed point theorems for α-ψ–expansive mappings in digital metric spaces and give some examples in support of our results.*

**Keywords and phrases**: *Fixed point, Digital topology, Digital contraction, Expansive Mappings, digital α-ψ–expansive mappings.*

**2010 Mathematical Subject Classification**: *47H10, 54E35, 68U10.*

## 1. INTRODUCTION

Topology is the study of geometric properties that does not depend only on the exact shape of the objects, but rather it acts on how the points are connected to each other. Infact, topology deals with those properties that remain invariant under the continuous transformation of a map. In 1979, Rosenfeld [12] introduced the concept of Digital Topology. Digital topology is concerned with geometric and topological properties of digital image. The digital images have been used in computer sciences (image processing and computer graphics). Digital topology also provides a mathematical basis for image processing operation in 2D and 3D digital images.For more detail, one can refer to [1, 7, 11].

In topology, infinitely many points are considered in arbitrary small neighborhood of a point but digital topology is concerned with finite number of points in a neighbourhood of a point. Therefore, one can distinguish easily between general topology and digital topology by considering the neighbourhood of a point as shown in the folowing figures.

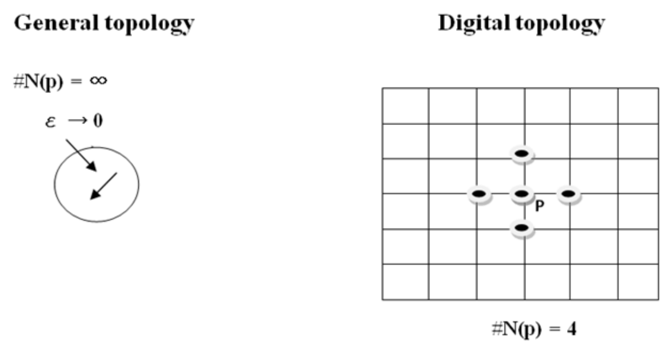



**FIGURE 1: Neighboorhood in General and Digital topology**

Digital image processing is a rapidly growing discipline with many applications in business (document reading), industry (automated assembly and inspection), medicine (radiology, haematology, etc.), and the environmental sciences (metrology, geology, land use management, etc.) and among many other fields. The work involves the analysis of picture i.e., the regions of which it is composed. A picture is input to the computer by sampling its brightness values at a discrete grid of points and digitizing or quantizing these values into binary digits. The result of this process is called a digital picture; it is a rectangular array of discrete values. The elements of this array are called pixels and the value of a pixel is called its gray level. The process of decomposing a picture into regions is called segmentation. Segmentation is basically a process of assigning the pixels. The one simple way of doing this process is called thresholding.

Once a picture has been segmented into subsets then it can be described by properties of subsets. Some of these properties depend on the gray levels of the points and some on the positions of the points. Basically, digital topology involves the concept of adjacency (surrounding) but not size or shape. The adjacency relations among the regions can be compactly represented by a graph. The two nodes of a graph are joined by an arc iff those two regions are adjacent.

## 2. TOPOLOGICAL VIEW POINT OF DIGITAL METRIC SPACES

Let $\mathbb{Z}^n$, $n \in \mathbb{N}$, be the set of points in the Euclidean $n$ dimensional space with integer coordinates.

**Definition 2.1.** [4] Let $l$, n be positive integers with $1 \leq l \leq$ n. Consider two distinct points

$p = (p_1, p_2, \dots p_n)$, $q = (q_1, q_2, \dots q_n) \in \mathbb{Z}^n$

The points p and q are $k_l$-adjacent if there are at most $l$ indices $i$ such that $|p_i - q_i| = 1$, and for all other indices $j$, $|p_j - q_j| \neq 1$, $p_j = q_j$.

(i) Two points $p$ and $q$ in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$ (see Figure 2).

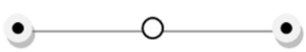


**Figure 2 2-adjacency**

(ii) Two points $p$ and $q$ in $\mathbb{Z}^2$ are 8-adjacent if the points are distinct and differ by at most 1 in each coordinate i.e., the 4-neighbors of $(x, y)$ are its four horizontal and vertical neighbors $(x \pm 1, y)$ and $(x, y \pm 1)$.

(iii) Two points $p$ and $q$ in $\mathbb{Z}^2$ are 4-adjacent if the points are 8-adjacent and differ in exactly one coordinate i.e., the 8-neighbors of $(x, y)$ consist of its 4-neighbors together with its four diagonal neighbors $(x + 1, y \pm 1)$ and $(x - 1, y \pm 1)$. (see Figure 3).

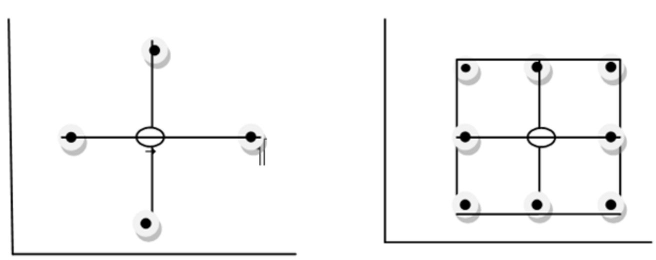


**FIGURE 3. 4-adjacency and 8-adjacency**

(iv) Two points $p$ and $q$ in $\mathbb{Z}^3$ are 26-adjacent if the points are distinct and differ by at most 1 in each coordinate. i.e.,

(a) Six face neighbours $(x \pm 1, y, z), (x, y \pm 1, z)$ and $(x, y, z \pm 1)$

(b) Twelve edge neighbours $(x \pm 1, y \pm 1, z), (x, y \pm 1, z \pm 1)$

(c) Eight corner neighbours $(x \pm 1, y \pm 1, z \pm 1)$

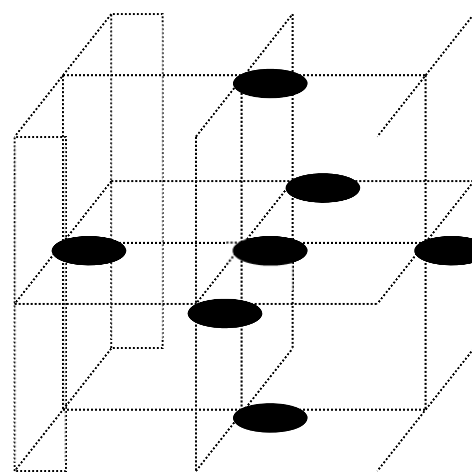
(v) Two points $p$ and $q$ in $\mathbb{Z}^3$ are 18-adjacent if the points are 26-adjacent and differ by at most 2 coordinate. i.e.,

(a)Twelve edge neighbours $(x \pm 1, y \pm 1, z), (x, y \pm 1, z \pm 1)$

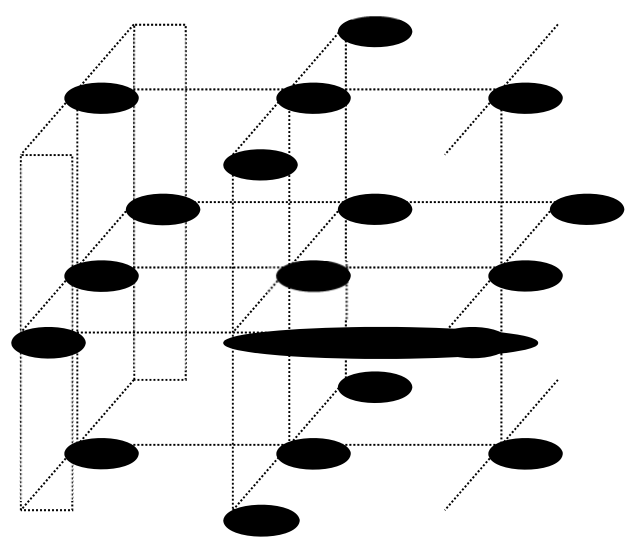
(b) Eight corner neighbours $(x \pm 1, y \pm 1, z \pm 1)$

(vi) Two points $p$ and $q$ in $\mathbb{Z}^3$ are 6-adjacent if the points are 18-adjacent and differ in exactly one coordinate. i.e.,

(a) Six face neighbours $(x \pm 1, y, z), (x, y \pm 1, z)$ and $(x, y, z \pm 1)$ (See Figure 4).



**6-adjacency**



**18-adjacency**

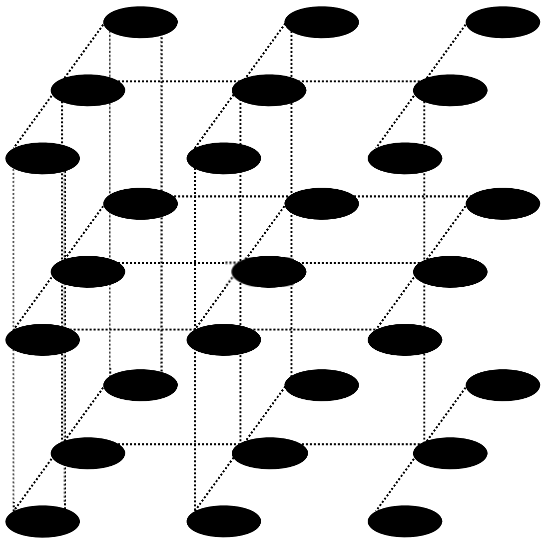


**FIGURE. 4. Adjacencies in $\mathbb{Z}^3$**

One can easily note that the coordination number of Na in the crystal structure of NaCl is 6 which is equal to adjacency relation in digital images of figure 5.

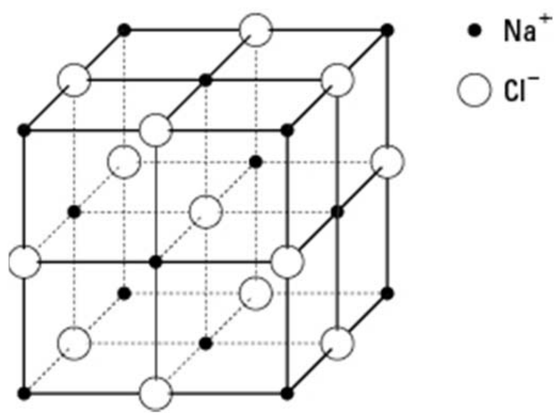



**FIGURE 5. Crystal structure of NaCI**

**Definition 2.2.** Let $\emptyset \neq X \subset \mathbb{Z}^n$, $n \in N$. A digital image is a pair $(X,\ k)$, where $k$ is an adjacency relation on $X$. Technically, a digital image $(X, k)$ is an undirected graph whose vertex set is the set of members of $X$ and whose edge set is the set of unordered pairs $\{x_{0,}x_1\} \subset X$ such that $x_0 \neq x_1$ and $x_0$ and $x_1$ are $k-$adjacent.

The notion of digital continuity in digital topology was developed by Rosenfeld [13] to study 2D and 3D digital images. Boxer [2] gives the digital version of several notions of topology and Ege and Karaca [5] described the digital continuous functions.

Let ℕ and ℝ denote the sets of natural numbers and real numbers, respectively. Boxer [3] defined a $k$ – neighbor of a point $p \in \mathbb{Z}^n$.

A $k$ – neighbor of a point $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is $k$ - adjacent to $p$, where $k \in \{2,4,6\ 8,18,26\}$ and $n \in \{1, 2, 3\}$.

The set $N_k\ (p) = \{q \mid q$ is $k$ - adjacent to $p\}$ is called the $k$ - neighborhood of $p$.

Boxer [2] defined a digital interval as

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}, \text{where } a, b \in \mathbb{Z} \text{ and } a < b.$$

A digital image $X \subset \mathbb{Z}^n$ is $k$-connected [8] if and only if for every pair of distinct points $x, y \in X$, there is a set $\{x_{0,}x_{1,}x_{2\ ,\dots,}x_r\}$ of points of a digital image $X$ such that $x = x_0$, $y = x_r$ where $x_i$ and $x_{i+1}$ are $k$ -neighbors and $i$ = 0, 1... $r$- 1.

**Definition 2.3.** Let $(X,k_0) \subset \mathbb{Z}^{n_0}$, $(Y, k_1) \subset \mathbb{Z}^{n_1}$ be digital images and f: $X \rightarrow Y$ be a function.

(i)If for every $k_0$-connected subset $U$ of $X$, $f\ (U)$ is a $k_1$-connected subset of $Y$, then $f$ is said to be $(k_0,k_1)$-continuous [3].

(ii) $f$ is $(k_0,k_1)$-continuous for every $k_0$-adjacent points $\{x_{0,}x_1\}$ of $X$, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are $k_1$-adjacent in $Y$[3] .

(iii) If $f$ is $(k_0,k_1)$-continuous, bijective and $f^{-1}$ is $(k_0,k_1)$-continuous, then $f$ is called $(k_0,k_1)$-isomorphism and denoted by $\cong_{(k_0,k_1)} Y$.

Now we start with digital metric space$(X, d, k)$ where $d$ is usual Euclidean metric on $\mathbb{Z}^n$ and $k$ denote the adjacency relation among the points in $\mathbb{Z}^n$.

**Definition 2.4.** [5] Let $(X, k)$ be a digital images set. Let $d$ be a function from $(X, k) \times (X, k) \rightarrow \mathbb{Z}^n$ satisfying all the properties of a metric space. The triplet $(X, d, k)$ is called a digital metric space.

**Proposition 2.5.** [7] Let $(X, d, k)$ be a digital metric space. A sequence $\{x_n\}$ of points of a digital metric space $(X, d, k)$ is

(i) a Cauchy sequence if and only if there is $\alpha \in \mathbb{N}$ such that for all, $\gneqq \alpha$ , then

$$d(x_n\ , x_m) \ \lneqq\ 1 \text{ i.e., } x_n = x_m.$$

(ii) convergent to a point $l \in X$ if for all $\epsilon \gneqq 0$, there is $\alpha \in \mathbb{N}$ such that for all $n \gneqq \alpha$ then $d(x_n, l) \lneqq \epsilon$, $i.e. x_n = l$.

**Proposition 2.6.** [7] A sequence $\{x_n\}$ of points of a digital metric space $(X, d, k)$ converges to a limit $l \in X$ if there is $\alpha \in \mathbb{N}$ such that for all $\gneqq \alpha$, then $x_n = l$.

**Theorem 2.7.** [7] A digital metric space $(X, d, k)$ is complete.

**Definition 2.8.** [5] Let $(X, d, k)$ be any digital metric space. A self map $f$ on a digital metric space is said to be digital contraction, if there exists a λ∈ [0, 1) such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda\, d(x, y)$$

**Proposition 2.9.** [5] Every digital contraction map $f: (X, d, k) \rightarrow (X, d, k)$ is digitally continuous.

**Proposition 2.10.** [7] In a digital metric space $(X, d, k)$, consider two points $x_i, x_j$ in a sequence $\{x_n\} \subset X$ such that they are $k$ -adjacent. Then they have the Euclidean distance $d(x_i, x_j)$ which is greater than or equal to 1 and at most √ t depending on the position of the two points.

**Definition 2.11.**[14] Let $(X, d)$ be a metric space and let $T: X \rightarrow X$ be a given mapping. We say that $T$ is an $\alpha - \psi$-contractive mapping if there exist two functions $\alpha: X \times X \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$

**Definition 2.12.** [14] Let $\Psi$ denote the family of all functions $\psi: [0, \infty) \rightarrow [0, \infty)$ which satisfy the following:

(i) $\psi^n(t) < \infty$ for each $t > 0$, where $\psi^n$ is the nth iterate of $\psi$;

(ii) $\psi$ is non-decreasing.

(iii) $\psi(0) = 0$.

**Remark 2.13**. (1) If: $\psi$: [0,∞) → [0,∞) is nondecreasing such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, then $\psi \in \Psi$.

(2) If $\psi$: [0,∞) → [0,∞) is upper semi continuous such that $\psi(t) < t$ for all $t > 0$, then $\lim_{n\rightarrow\infty} \psi^n(t) = 0$ for all $t > 0$.

**Main Result**

**3. DIGITAL α-ψ– EXPANSIVE MAPPINGS AND COMMON FIXED POINTS**

Now we introduce the concept of digital-α-ψ–expansive mappings in Digital metric spaces analogue to the α-ψ–expansive mappings in metric spaces.

**Definition 3.1.** Let $(X, d, k)$ be a digital metric space and $S: X \rightarrow X$ be a given mapping. We say that S is a digital-α-ψ–expansive mapping, if there exists two functions $\alpha: X \times X \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y$ in $X$, we have

$$\psi(d(Sx, Sy)) \geq \alpha(x, y)\, d(x, y).$$

**Remark 3.2.** Clearly, any expansive mapping is a digital-α-ψ–expansive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, for all $t \geq 0$ and $k \in (0, 1)$.

**Definition 3.3.** Let $S: X \rightarrow X$ and $\alpha: X \times X \rightarrow [0, \infty)$. We say that $S$ is $\alpha$-admissible if for all $x, y$ in $X$, we have $\alpha(x, y) \geq 1 \Longrightarrow \alpha(Sx, Sy) \geq 1$.

Now, we prove our main results.

**Theorem 3.4.** Let $(X, d, k)$ be a complete digital metric space and $S: X \rightarrow X$ be a bijective, digital-α-ψ–expansive mapping satisfies the following conditions:

(i) $S^{-1}$ is $\alpha$-admissible;

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, S^{-1}x_0) \geq 1$;

(iii) $S$ is digitally continuous.

Then $S$ has a fixed point, that is, there exists $u$ in $X$ such that $Su = u$.

**Proof.** Let us define the sequence $\{x_n\}$ in $X$ by $x_n = Sx_{n+1}$ for all $n \geq 0$, where $x_0 \in X$ is such that $\alpha(x_0, S^{-1}x_0) \geq 1$. If $x_n = x_{n+1}$ for some $n$, then $x_n$ is a fixed point of $S$. So, we can assume that $x_n \neq x_{n+1}$ for all $n$.

It is given that $\alpha(x_0, x_1) = \alpha(x_0, S^{-1}x_0) \geq 1$. (1)

Recalling that $S^{-1}$ is $\alpha$-admissible, we have, $\alpha(S^{-1}x_0, S^{-1}x_1) = \alpha(x_1, x_2) \geq 1$.

Continuing in this way, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n = 0, 1, 2, \ldots \quad (2)$$

From (1) and (2), it follows that for all $n \geq 1$, we have $d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n+1})\, d(x_n, x_{n+1}) \leq \psi(d(Sx_n, Sx_{n+1})) = \psi(d(x_{n-1}, x_n))$.

Since $\psi$ is non-decreasing, by induction, we have

$d(x_n, x_{n+1}) \le \psi^n(d(x_{n-1}, x_n)$ for all $n \ge 1$. (3)

Using (3), we have $d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \le \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \le \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1))$.

Since, $\psi \in \Psi$ and $d(x_0, x_1) > 0$, we get $\sum_{k=0}^{\infty} \psi^k(d(x_0, x_1)) < \infty$.

Thus, we have $\lim_{n\to\infty} d(x_n, x_m) = 0$.

This implies that $\{x_n\}$ is a digital Cauchy sequence in digital metric space $(X, d, \kappa)$. But $(X, d, \kappa)$ is complete, so there exists $u$ in $X$ such that $x_n \to u$ as $n \to \infty$. From the continuity of S, it follows that $x_n = Sx_{n+1} \to Su$ as $n \to \infty$. By the uniqueness of the limit, we get $u = Su$, that is, $u$ is a fixed point of S.

In what follows, we prove that Theorem 3.4 still holds even though S is not necessarily continuous, assuming the following condition:

(M) If $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n$ and $\{x_n\} \to x \in X$ as $n \to \infty$, then

$\alpha(S^{-1}x, S^{-1}x_n) \ge 1$, for all $n$. (4)

**Theorem 3.5.** If in Theorem 3.4, we replace the continuity of $S$ by the condition (M), then the result holds true.

**Proof.** Following the proof of Theorem 3.2, we know that $\{x_n\}$ is a digital Cauchy sequence in $X$ such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n$ and $x_n \to u$ as $n \to \infty$. Now, from the hypothesis (4), we have

$\alpha(S^{-1}u, S^{-1}x_n) \ge 1$, for all $n$. (5)

Using (1) and (5), we get

$d(S^{-1}u, u) \le d(S^{-1}u, x_{n+1}) + d(x_{n+1}, u) = d(S^{-1}u, S^{-1}x_n) + d(x_{n+1}, u) \le \alpha(S^{-1}u, S^{-1}x_n)\, d(S^{-1}u, x_{n+1}) + d(x_{n+1}, u) \le \psi(d(u, x_n)) + d(x_{n+1}, u)$.

Continuity of $\psi$ at $t = 0$ implies that $d(S^{-1}u, u) = 0$ as $n \to \infty$. That is, $S^{-1}u = u$. Consider, $Su = S(S^{-1}u) = u$, which implies that, $u$ is a fixed point of S.

We now present some examples in support of our results and show that the hypotheses in Theorems 3.2 and 3.3 do not guarantee uniqueness.

To ensure the uniqueness of the fixed point in Theorems 3.2 and 3.3, we consider the condition:

(S): For all $u, v \in X$, there exists $w \in X$ such that $\alpha(u, w) \ge 1$ and $\alpha(v, w) \ge 1$.

**Theorem 3.6.** Adding the condition (S) to the hypotheses of Theorem 3.4 (resp. Theorem 3.5), we obtain the uniqueness of the fixed point of S.

**Proof.** From Theorem 3.4 and 3.5, the set of fixed points is non-empty. We shall show that if $u$ and $v$ are two fixed points of $S$, that is, $S(u) = u$ and $S(v) = v$, then $u = v$.

From the condition (S), there exists $w \in X$ such that

$\alpha(u, w) \ge 1$ and $\alpha(v, w, w) \ge 1$. (6)

Recalling the $\alpha$-admissible property of $S^{-1}$, we obtain from (6)

$\alpha(u, S^{-1}w) \ge 1$ and $\alpha(v, S^{-1}w) \ge 1$. (7)

Therefore, by repeatedly applying the $\alpha$-admissible property of $S^{-1}$, we get

$\alpha(u, S^{-n}w) \ge 1$ and $\alpha(v, S^{-n}w) \ge 1$, for all $n$ in $\mathbb{N}$. (8)

From the inequalities (1) and (8), we get $d(u, S^{-n}w) \le \alpha(u, S^{-n}w)d(u, S^{-n}w) \le \psi(d(Su, S^{-n+1}w) = \psi(d(u, S^{-n+1}w))$.

Repetition of the above inequality implies that $d(u, S^{-n}w) \le \psi(d(u, w))$, for all $n \in \mathbb{N}$. Thus, we have $S^{-n}w \to u$ as $n \to \infty$.

Using the similar steps as above, we obtain $S^{-n}w \to v$ as $n \to \infty$. Now, the uniqueness of the limit of $S^{-n}w$ gives $u = v$. This completes the proof.

**Example 3.7.** Let us consider the digital metric space $(X, d, k)$, with the digital metric defined by $d(x, y) = 2|x - y|$. Consider the self mapping $S: X \to X$ given by

$$T(x) = \begin{cases} 2x - \frac{11}{6}, for\ x > 1 \\ \frac{x}{6}, for\ x \le 1 \end{cases}.$$

and $\alpha: X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 0 \; if \; x,y \in [0,1) \\ 1 \; otherwise \end{cases}$$

Clearly, $S$ is a digital-α-ψ–expansive mapping with $\psi(a) = \frac{a}{6}$ for all $a \geq 0$. In fact for all $x, y \in X$, we have $d(Sx, Sy) \geq \alpha(x,y)\, d(x,y)$.

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, S^{-1}x_0) \geq 1$. In fact, for $x = 1$, we have $\alpha(1, S^{-1}1) = 1$. Obviously, $S$ is digitally continuous, and so it remains to show that $S^{-1}$ is $\alpha$-admissible. For this, let $x, y \in X$ such that $\alpha(x,y) \geq 1$. This implies that $x \geq 1$ and $y \geq 1$, and by the definitions of $S^{-1}$ and $\alpha$, we have $S^{-1}(x) = \frac{x}{2} + \frac{11}{12} \geq 1$, $S^{-1}(y) = \frac{y}{2} + \frac{11}{12} \geq 1$ and $\alpha(S^{-1}x, S^{-1}y) = 1$. Then $S^{-1}$ is $\alpha$-admissible. Now, all the hypothesis of Theorem 3.4 are satisfied. Consequently, $S$ has a fixed point. Clearly, $x = 0$ and $x = \frac{11}{6}$ are two fixed points of $S$.

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